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# Present Value Analysis of Traditional Loans

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# Present Value Analysis of Traditional Loans

## Introduction

The use of loans in the analysis of cash flows is critical as the interest is a deductable expense and the amount is needed for the after-tax cash flow analysis. Purchases of expensive capital equipment utilize loans as the cash flows after taxes usually give better results for utilizing loans than paying for the purchase entirely out of retained earnings. The use of loans allows corporations, as well as individuals, to make purchases sooner and are a driving force used to improve economic conditions. The government has reduced their interest rates to nearly zero in an attempt to improve economic activity.

The calculation of the amounts of interest and principal are somewhat difficult for students and several of the newer textbooks no longer give the formulas used to make the calculations, but instead give an Excel expressions to calculate the interest and principal values. An understanding of the calculations is important before one uses the expressions so one can validate the expressions being used as well as develop their own models. The students only insert numbers in the requested format and get answers, but have no intuition as to what is occurring in the process. The present value analysis can give students a better understanding of what is occurring during the calculation of the principal and interest values of a loan. The current or traditional approach of calculating the principal and interest values will be presented and then followed by the present value approach.

Students who use software calculation packages such as Excel cannot be proficient in using it unless they understand how the calculations are being made. It is important that students have enough knowledge about the calculations to ascertain that they have entered the input values correctly and know the range of the output expected.

## Traditional Approaches to Calculations of Interest and Principal Portions of Loan Payments.

The loans considered will be traditional loans where equal payment amounts are considered for each period. The items to be considered are the amount of the payment, the amount of interest of the payment, the amount of principal of the payment, the total amount of principal remaining which is also known as the unpaid balance, the total principal that has been paid to time "t", and the total interest that has been paid to time "t". The symbols used for these items are expressed in Table 1.

The traditional approach for calculating the various expressions starts with the calculation of the unpaid balance of the loan. However, there are two different approaches, one utilizing a present worth approach and the other using a future worth approach. Both methods will be considered, starting with the present worth approach.

Table 1. Nomenclature Used in Formulas Presented

| Symbol | Description of Symbol |
|---|---|
| LV | Total initial amount of the loan |
| A | Annual end-of-period loan payment |
| n | Number of periods for loan payments |
| t | Specific time period of interest |
| i | Amount of Interest Rate per period |
| P(t) | Amount of Principal paid in period t |
| I(t) | Amount of Interest paid in period t |
| PT(t) | Total amount of Principal paid in periods 1 through t |
| IT(t) | Total amount of Interest paid in periods 1 through t |
| U(t) | Amount of Principal to be paid in remaining periods, also known as Unpaid Balance |
| PV(P) | Present Value of the Amount of Principal paid in period t |

A. Unpaid Balance - Present Worth Approach

The unpaid balance present worth approach is based upon the amount of principal to be paid during the remaining life of the loan which is represented as U(t) in the symbols listed in Table 1. If the amount of the periodic payment is A, then the amount of principal to be paid in the remaining life of the loan during the remaining n-t periods can be expressed as the present worth of the remaining payments of the loan as:

$$U(t) = A \times [P/A,i,(n-t)] \qquad (1)$$

Using the formulas for the [P/A, i, (n-t)] one obtains:

$$U(t) = A \times [((1 + i)^{(n-t)} - 1)] / [ i \times (1 + i)^{(n-t)}] \qquad (2)$$

The interest amount per period can be determined as the interest rate times the unpaid balance of the previous period where t is 1 or higher, that is:

$$I(t) = i \times U(t-1) \qquad (3)$$

$$I(t) = i \times A \times [P/A,i,(n-(t-1))] \qquad (4)$$

Using the formula for [P/A,i,(n-(t-1))] one obtains the expressions

$$I(t) = i \times A \times [((1 + i)^{(n-(t-1))} - 1)] / [ i \times (1 + i)^{(n-(t-1))}]$$

which can be reduced to

$$I(t) = A \times [((1 + i)^{(n-(t-1))} - 1)] / [ (1 + i)^{(n-(t-1))}] \qquad (5)$$

The principal per period, P(t), can be determined from A and the interest per period I(t) by:

$$P(t) = A - I(t) \qquad (6)$$

$= A - i \times A \times [P/A,i,(n-(t-1))]$

$= A \times ( 1- i \times [P/A,i,(n-(t-1))] )$

since $( 1- i \times [P/A,i,(n-(t-1))] ) = [P/F,i,(n-(t-1))]$

thus $P(t) = A \times [P/F,i,(n-(t-1))]$ (7)

and using the formula for [P/F,i,(n-(t-1))] one obtains:

$$P(t) = A \times (1+i)^{(n-(t-1))} \quad (8)$$

The total principal paid through t periods can be represented as:

$$PT(t) = LV - U(t) \quad (9)$$

$$= LV - A \times [P/A,i,(n-t)] \quad (10)$$

and using the formula for [P/A,i,(n-t)] one obtains:

$$PT(t) = LV - A \times [((1 + i)^{(n-t)} - 1)] / [ i \times (1 + i)^{(n-t)}] \quad (11)$$

The total interest paid through t periods can be expressed as:

$$IT(t) = t \times A - PT(t) \quad (12)$$

$$= t \times A - LV + A \times [((1 + i)^{(n-t)} - 1)] / [ i \times (1 + i)^{(n-t)}] \quad (13)$$

The five expressions have exponents of n-t or n-(t-1) which often causes students difficulty in performing the calculations. The unpaid balance future worth approach eliminates some of this difficulty.

B. Unpaid Balance - Future Worth Approach

The unpaid balance future worth approach considers the unpaid balance to be the difference between the future worth of the initial loan and the future worth of the payments made on the loan at the time of interest t, that is:

$$U(t) = LV \times [F/P,i,t] - A \times [F/A,i,t] \quad (14)$$

if one uses the identity relationship that

$$[F/P,i,t] = i \times [F/A,i,t] + 1$$

then Equation (14) becomes

$$U(t) = LV \times [i \times [F/A,i,t] + 1] - A \times [F/A,i,t]$$

$$= LV - (A - i \times LV) \times [F/A,i,t] \qquad (15)$$

and using the formula for [F/A,i,t] one obtains:

$$U(t) = LV - (A - i \times LV) \times [((1+i)^t - 1) / i] \qquad (16)$$

The total principal can determined from the difference between the loan amount and the unpaid balance at time t, that is:

$$PT(t) = LV - U(t) \qquad (17)$$

$$= LV - LV - (A - i \times LV) \times [F/A,i,t]$$

$$PT(t) = (A - i \times LV) \times [F/A,i,t] \qquad (18)$$

and using the formula for [F/A,i,t] one obtains:

$$PT(t) = (A - i \times LV) \times [((1+i)^t - 1) / i] \qquad (19)$$

The total interest can be determined from the difference between the total amount of the payments for the t periods and the total principal for the that time as indicated by:

$$IT(t) = txA - PT(t) \qquad (20)$$

$$IT(t) = txA - (A - i \times LV) \times [F/A,i,t] \qquad (21)$$

and using the formula for [F/A,i,t] one obtains:

$$IT(t) = txA - (A - i \times LV) \times [((1+i)^t - 1) / i] \qquad (22)$$

The principal per period can be determined from the difference between the total principal for t periods minus the total principal for (t-1) periods and this is shown as:

$$P(t) = PT(t) - PT(t-1) \qquad (23)$$

$$P(t) = (A - i \times LV) \times [F/A,i,t] - (A - i \times LV) \times [F/A,i,(t-1)] \qquad (24)$$

and using the formula for [F/A,i,t] one obtains:

$$= (A - i \times LV) \times [((1+i)^t - 1) / i] - (A - i \times LV) \times [((1+i)^{(t-1)} - 1) / i]$$

$$= (A - i \times LV) \times [((1+i)^t - (1+i)^{t-1}) / i]$$

$$= (A - i \times LV) \times (1+i)^{t-1} \qquad (25)$$

This expression can be written in terms of only t as:

$$P(t) = ((A - i \times LV)/(1+i)) \times (1+i)^t \quad (26)$$

The expression for the interest per period can be determined by the difference between the payment and the principal part of the payment for that period as:

$$I(t) = A - P(t) \quad (27)$$

$$I(t) = A - (A - i \times LV)(1+i)^{t+1} \quad (28)$$

or as $$I(t) = A - ((A - i \times LV) /(1+i)) (1+i)^t \quad (29)$$

Thus it is possible to develop expressions using the powers in terms on only t and avoid the n-t and (n-(t-1)) terms which often lead to confusion.

Both methods are based upon starting with the unpaid balance and although the principal amounts increase and interest amounts decrease with the additional payments, there is little insight as to what is happening in the loan process. The values for a loan of $ 10,000 for 10 years at an interest rate of 10% are presented in Table 2.

Table 2. Loan Parameters and Results for Unpaid Balance, Total Principal Paid, Total Interest Paid, Principal per Period and Interest per Period.

| Loan Value LV | Interest Rate i | Loan Life n | Loan Payment A |
|---|---|---|---|
| 10000 | 0.1 | 10 | 1627.45 |

| Period t | Unpaid Balance U(t) | Total Principal Paid PT(t) | Total Interest Paid IT(t) | Principal per Period P(t) | Interest per Period I(t) |
|---|---|---|---|---|---|
| 0 | 10000.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1 | 9372.55 | 627.45 | 1000.00 | 627.45 | 1000.00 |
| 2 | 8682.35 | 1317.65 | 1937.25 | 690.20 | 937.25 |
| 3 | 7923.13 | 2076.87 | 2805.49 | 759.22 | 868.23 |
| 4 | 7087.99 | 2912.01 | 3597.80 | 835.14 | 792.31 |
| 5 | 6169.33 | 3830.67 | 4306.60 | 918.66 | 708.80 |
| 6 | 5158.81 | 4841.19 | 4923.53 | 1010.52 | 616.93 |
| 7 | 4047.24 | 5952.76 | 5439.41 | 1111.57 | 515.88 |
| 8 | 2824.51 | 7175.49 | 5844.14 | 1222.73 | 404.72 |
| 9 | 1479.50 | 8520.50 | 6126.59 | 1345.00 | 282.45 |
| 10 | 0.00 | 10000.00 | 6274.54 | 1479.50 | 147.95 |
| Totals | | | | 10000.00 | 6274.54 |

## The Present Value Approach to Loan Analysis

During the teaching of the advanced engineering graduate students and a project was assigned which has the class making their own cash flow model which involved loans, depreciation, as well as the traditional revenues, expenses, taxes, and etc. The students had difficulty in determining the interest and principal amounts for the loans over the various periods and it was suggested they review their basic engineering textbook, but the textbooks did not have the formulas. Another part of the project considered issues of constant and current dollars and one method of checking the results was to calculate the present value as it should be the same. While doing this I decided to look at the present value of loans and found a very interesting discovery. The results for the present value analysis are presented in Table 3.

Table 3. Present Value Approach of Loan Payments

| Loan Value LV | Interest Rate i | Loan Life n | Loan Payment A |
|---|---|---|---|
| 10,000 | 0.1 | 10 | 1627.45 |

| Period t | Interest per Period I(t) | Principal per Period P(t) | PV Interest per Period PVI(t) | PV Principal per Period PVP(t) | Unpaid Balance U(t) | Total Interest Paid IT(t) | Total Principal Paid PT(T) |
|---|---|---|---|---|---|---|---|
| 0 | 0.00 | 0.00 | 0.00 | 0.00 | 10000.00 | 0.00 | 0.00 |
| 1 | 1000.00 | 627.45 | 909.09 | 570.41 | 9372.55 | 1000.00 | 627.45 |
| 2 | 937.25 | 690.20 | 774.59 | 570.41 | 8682.35 | 1937.25 | 1317.65 |
| 3 | 868.23 | 759.22 | 652.32 | 570.41 | 7923.13 | 2805.49 | 2076.87 |
| 4 | 792.31 | 835.14 | 541.16 | 570.41 | 7087.99 | 3597.80 | 2912.01 |
| 5 | 708.80 | 918.66 | 440.11 | 570.41 | 6169.33 | 4306.60 | 3830.67 |
| 6 | 616.93 | 1010.52 | 348.24 | 570.41 | 5158.81 | 4923.53 | 4841.19 |
| 7 | 515.88 | 1111.57 | 264.73 | 570.41 | 4047.24 | 5439.41 | 5952.76 |
| 8 | 404.72 | 1222.73 | 188.81 | 570.41 | 2824.51 | 5844.14 | 7175.49 |
| 9 | 282.45 | 1345.00 | 119.79 | 570.41 | 1479.50 | 6126.59 | 8520.50 |
| 10 | 147.95 | 1479.50 | 57.04 | 570.41 | 0.00 | 6274.54 | 10000.00 |
| Totals | 6274.54 | 10000.00 | 4295.87 | 5704.13 | | | |

The obvious factor is that the present value of the payments on the principal are constant per period throughout the life of the loan and that is what occurs during the life of the loan. A loan is a series of equal payments of the present value of the principal, so the actual amounts of the principal can be determined from the present value very easily. Also, the present value is easily obtained from the loan amount, principal and interest rate.

$$PV(P) = (A - i \times LV)/(1+i) \quad (30)$$

This is a relatively simple expression and the value is the same for all periods which is the unique in the analysis of loan principal analysis. The other expressions can be obtained from variations of the present value of the principal payment. Another interesting observation is that the sum of the present value amounts per period of the interest and the sum of the present value amounts of the principal total to be the initial value of the loan value. This can lead to further consideration in the determination of the interest rate at which the present worth of the interest payments exceed the present worth of the principal payments.

The principal per period can be determined by as a nice, easy to explain expression, as:

$$P(t) = PV(P) \times (1+i)^t \tag{31}$$

Verbally, this can be stated as taking the present worth of the principal and adjusting it to the period of interest, that is

$$P(t) = PV(P) \times (F/P,i,t) \tag{32}$$

The interest per period can be determined easily from the loan payment minus the principal for the period, that is:

$$I(t) = A - P(t) \tag{33}$$

or in terms of PV(P) as

$$I(t) = A - PV(P) \times (1+i)^t \tag{34}$$

The total principal paid up to period t can be calculated using the geometric series to sum the P(t) values which results in:

$$PT(t) = \sum_{1}^{t} P(t) \tag{35}$$

$$PT(t) = \sum_{1}^{t} PV(P) \times (1 + i)^t \tag{36}$$

$$PT(t) = PV(P) \times (1+i) \times [((1+i)^t - 1)/i] \tag{37}$$

Verbally, this can be stated as summing the equivalent first period payment over the period of interest, that is

$$PT(t) = PV(P) \times (1+i) \times (F/A,i,t) \tag{38}$$

This could also be rewritten as:

$$PT(t) = P(t=1) \times (F/A,i,t) \tag{39}$$

The unpaid balance of the loan at any period can be calculated from the total amount loan minus the total principal paid at time period t.

$$U(t) = LV - PT(t) \tag{40}$$

$$U(t) = LV - PV(P) \times (1 + i) \times [((1 + i)^t - 1)/i] \tag{41}$$

$$U(t) = LV - PV(P) \times (1+i) \times (F/A,i,t)$$

The total interest paid up to period t can be calculated from the total amount of the payments at time t and the total principal at time t via:

$$IT(t) = t \times A - PT(t) \tag{41}$$

$$IT(t) = t \times A - PV(P) \times (1 + i) \times ((1 + i)^t - 1)/i \tag{42}$$

or

$$IT(t) = t \times A - PV(P) \times (1+i) \times (F/A,i,t) \tag{43}$$

Note that all of the expressions can be expressed in terms of PV(P) and that the expressions do not need the [n-t] or [n-(t-1)] terms which often confuse the students.

**Preliminary Results**

Students in the Advanced Engineering Economics class were given the assignment to solve for the interest, principal, unpaid balance, and total principal for each period for a loan using the three methods and calculating the present worth of the principal and interest for the present value approach. They were asked to indicate which method they preferred and why, and several students did the calculations but did not discuss what method they preferred.

The students preferred The Present Value Approach(60%), second was The Unpaid Balance - Present Worth Approach(30%) and third was The Unpaid Balance - Future Worth Approach(10%). The comments by the students were:

The Present Value Approach(60%)

- total principal and total interest were equal to the loan value
- calculations were easier
- easier to calculate principal amount per period
- easier to calculate, better to understand, present value of payments for interest and principal equal loan value, and don't to calculate unpaid balance to find principal for period
- calculations are easier and other methods were easier to make mistakes
- can calculate more information and easier

The Unpaid Balance - Present Worth Approach(30%)

- all methods easy, bet to unpaid balance quickly
- easy to follow logic of method
- more useful to know the present amount to make decisions about the future

The Unpaid Balance - Future Worth Approach(10%)
- a loan cannot be considered as an investment

**Conclusions**

The present value analysis of a loan indicates what a loan represents; it is the payment of equal amounts of principal from the start date of the loan. The payments are done at the end of the period instead of the beginning of the period this is why the values appear different. The value of the principal payments increase directly in proportion to the future worth factor. The observation that the present value of the interest payment is the same removes a lot of the mystery about the amounts of principal and interest in loan calculations. The basic expressions can be summarized as:

| | | | |
|---|---|---|---|
| Present Worth of Principal | $PV(P)$ | $= (A - i \times LV) / (1+i)$ | (44) |
| Principal for Period t | $P(t)$ | $= PV(P) \times (F/P,i,t)$ | (45) |
| Interest for Period t | $I(t)$ | $= A - P(t)$ | (46) |
| Total Principal Paid to Period t | $PT(t)$ | $= PV(P) \times (1+i) \times (F/A,i,t)$ | (47) |
| Total Interest Paid to Period t | $IT(t)$ | $= t \times A - PT(t)$ | (48) |
| Unpaid Balance at Period t | $U(t)$ | $= LV - PT(t)$ | (49) |

This approach will help students understand the philosophy of a loan with constant present value principal payments and make the calculations much easier to perform. This approach of calculating the various components in a loan and especially the interest per period should make cash flow analysis with loans much easier for the students to calculate. Preliminary testing indicates students prefer this new approach.