A General Structured Procedure to Solve Machine Design Problems

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Abstract

This paper presents a general structured procedure using eight steps to solve machine design problems. The design of a circular shaft subjected to combined loading is presented to show the general structured approach. All equations are formulated symbolically and solved using a modern engineering tool. One significant advantage is that we can solve symbolic equations for any variable value. Therefore, the design process generally requires solving problems over a range of variable values to obtain a satisfactory design. An essential part of our approach is educating our students to question, test, and verify "answers" to all of their problem solutions. Verifying answers is done by developing and implementing test case scenarios to verify the problem's validity. Lecture examples and homework problems throughout the course are solved with all equations formulated symbolically, and test cases are used to verify the equations. The circular shaft design project is divided into multiple phases. Each phase of the project requires the material covered up to a point in the course. A project phase is assigned once a topic is covered in the lecture and reinforced through homework and quizzes.

Introduction

Engineering design, defined by ABET [1], "is a process of devising a system, component, or process to meet desired needs and specifications within constraints. It is an iterative, creative decision-making process in which the basic sciences, mathematics, and engineering sciences are applied to convert resources into solutions. Engineering design involves identifying opportunities, developing requirements, performing analysis and synthesis, generating multiple solutions, evaluating solutions against requirements, considering risks, and making trade-offs to obtain a high-quality solution under the given circumstances. For illustrative purposes only, examples of possible constraints include accessibility, aesthetics, codes, constructability, cost, ergonomics, extensibility, functionality, interoperability, legal considerations, maintainability, manufacturability, marketability, policy, regulations, schedule, standards, sustainability, or usability."

The authors believe that machine design textbooks make it a challenge to carry out an "iterative, creative, decision-making process in which the basic sciences, mathematics, and engineering sciences are applied to convert resources into solutions." This paper proposes a problem solution with a symbolic formulation approach to overcome this challenge to perform solution verification and parametric design studies.

The authors propose that all variables be retained symbolically, and all equations are written symbolically in natural form without any algebraic manipulation. Once all equations are developed, they are solved by the method of choice, i.e., by hand and/or, preferably, a modern engineering tool. The authors strongly endorse using a commercial program equation solver for all but the most straightforward problems, supported by verifying the result. This approach allows the students to focus on the basic fundamental physics of the problem rather than on the algebraic manipulation required to isolate the required solution variable(s). The proposed approach allows a natural extension to design since all equations are in symbolic form. With equations written in symbolic form, they are entered into a modern engineering tool (equation solver) and validated. The equations then may be used for the repetitive analysis of a structure and design of a similar structure. We must select the dimensions and materials for a given loading. Incorporating a computer equation solver with the 'raw' fundamental symbolic equations leads to easy design applications. It has the added benefits of reduced opportunity for algebraaic errors and increased engineering productivity.

We can use the approach proposed in this paper can be used to demonstrate ABET criterion 3.2 [1] that states "an ability to apply engineering design to produce solutions that meet specified needs …," applied to individual structural components. Furthermore, we can use the approach in other courses, i.e., statics, mechanics of materials, structural design, etc. In that case, we can use this approach demonstrate ABET Criteria 5d [1] "culminating major engineer design experience that 1) incorporates appropriate engineering standards and multiple constraints, and 2) is based on the knowledge and skills acquired in earlier course work." The widely used machine design textbooks [2] – [5] do not use a symbolic approach as presented in this paper. Norton textbook [5] provides computer files for examples and case studies using the software cited in this paper that one could use to solve the problem symbolically.

General Structure Procedure to Solve Machine Design Problems

For the general problem involving deformation, our proposed non-traditional structured problem-solving format contains eight analysis steps. The students must follow the appropriate steps listed below for every in-class lecture, homework problem, and design project they solve based on reference [6] and [7].

1. *Model.* The success of any analysis is dependent on the validity and appropriateness of the model used to predict and analyze its behavior in a real system, whether centric axial loading, torsion, bending, or a combination of the above. Assumptions and limitations also need to be stated. This step is not explicitly emphasized in any mechanics of materials or machine design textbooks. All dimensions and forces are defined symbolically.

2. *Free-Body Diagrams.* This step is where all the free-body diagrams initially thought to be required for the solution are drawn. The free-body diagrams include the complete structure and/or parts of the structure. Very importantly, all dimensions and loads, even those which are known, are defined symbolically.

3. *Equilibrium Equations.* The equilibrium equations for each free-body diagram required for a solution are written. All equations are formulated symbolically. There is no attempt made at this point to isolate the unknown variables. However, we must examine every term in each equation for dimensional homogeneity.

4. *Deformation Equations.* The deformation formulas are written for each part of a structure based on the Model in Step 1 based on the method of segments [7]. All equations are formulated symbolically, and there is no algebraic manipulation. We must examine every term in each equation for dimensional homogeneity.

5. *Compatibility and Boundary Conditions.* One or more compatibility equations are written in symbolic form to relate the displacements. A compatibility diagram is used when appropriate to assist in developing the compatibility equations. All equations are formulated symbolically, and there is no algebraic manipulation. Every term in each equation must be examined for dimensional homogeneity. Although compatibility equations are commonly written for indeterminate problems, the authors emphasize their use for determinate problems just as is done in the mechanics of materials textbooks [8] – [10].

6. *Complementary and Supporting Formulas.* Steps 1 through 5 are sufficient to solve for the (primary) variables force and displacement in a structures problem. Step 6 includes complementary formulas for other (secondary) variables such as stress and strain, variables which may govern the maximum allowable in service values of force and displacement but do not affect the governing equilibrium or deformation equations. Supporting formulas might be required to supply variable values in the material law equations and complementary formulas; formulas such as area, the moment of inertia, centroid location of a cross-section, volume, etc.

   The complementary and supporting formulas are written symbolically and are necessary to develop a complete analysis. The complementary formulas might involve solution governing variables such as stress, strain, and stiffness. Supporting formulas may be necessary to completely define variables in Steps 3 through 5 and in the complementary formulas. These formulas might include a cross-sectional area, polar moment of inertia, centroid location, the moment of inertia, section modulus, effective length, the radius of gyration, etc.

7. *Solve.* The independent equations developed in Steps 3 through 6 solve the problem. The students compare the number of independent equations and the number of unknowns. The authors emphasize that the student should not proceed until the number of unknowns equals the number of independent equations.

   The solution may be obtained by hand, and this generally requires algebraic manipulation. Alternatively, the solution of any number of equations, linear or non-linear, can be achieved with a modern engineering tool. With the intelligent application of verification (Step 8), the computer program is a much more reliable calculation device than a calculator. The students are allowed to select the modern engineering tool of their choice, and this might include Mathcad [11], Matlab [12], and TKSolver [13]. The authors have not seen this solution procedure in any machine design or mechanics of materials textbook.

8. *Verify.* This critical step is a critique of the answer and is discussed in-depth in the next section. The paper [14] focuses on educating students to question, test, and verify problem solutions for mechanics of materials problems.

Statics problems require only Steps 1, 2, 3, 7, and 8, and mechanics of materials problems require Steps 1 to 8. Like our approach, the statics [15] and mechanics of materials [16] textbooks use the SMART problem-solving methodology, i.e., **S**trategy, **M**odeling, **A**nalysis, and **R**eflect and **T**hink. A significant difference is that this paper formulates all equations symbolically, and then then the unknowns are solved. Step 8 is also considered in the mechanics of materials textbook [8]. The authors are not aware of the proposed methodology being used in machine design textbooks. The authors are not aware of other structured problem-solving methods like those used in this paper and SMART in statics, mechanics of materials, and machine design textbooks.

Pedagogically, the step-by-step solution format presented allows students to build a structure in their minds of how to approach a problem and solve it efficiently. The authors believe that this step-by-step procedure will help students build logic, promote analytical thinking, provide an accurate physical understanding of the subject, and, hopefully, extend the same disciplined process to other courses.

Step 8 Verify: Question and Test to Verify The Answers

Step 5, Verify, may be new to students, but it is critical! As a professional, one must be prepared to guarantee their solution. Attempts at verification of the solution may take many forms, and although in some cases it may not yield absolute proof, it does improve the level of confidence. Verification might involve comparison with a hand calculation, comparison with solutions of similar problems, examining limiting and obvious solution cases, and comparison with experimental data.

One of our educational goals is to convince students of the wisdom to question and test solutions to verify their 'answers.' We do this by integrating verification as part of the general structured solution procedure. Verification is new to almost all undergraduate students, but it is critical and must be formally incorporated into the solution process! The power of our proposed use of the computer equation solver rests in the ability to quickly and easily run many cases to verify the problem solution. Once an answer has been confirmed, the computer model becomes an essential part of parametric studies and design.

Using Modern Engineering Tools for Design

Much of the problem-solving in formal academic courses involves analyzing a single modeled system; calculations, typically, are performed once. However, the engineer involved in the design is often confronted with many calculations of a repetitive nature. An initial concept of a structure, the "early design stage," requires initial sizing calculations. Refinement requires more calculations. Resizing to use standard commercially available parts requires more calculations, with the probably added complexity of swapping some variables from known to unknown. Any analysis tool that reduces the boredom of this process and, at the same time, reduces the risk of calculation error should be investigated.

There is three relatively popular engineering equation solving tools available in both professional and student form; Mathcad [11], MATLAB [12], and TK Solver [13]. A computer equation solving program is a much more reliable calculation device than a calculator. All three programs have technical computation problem-solving features far too numerous to discuss. For this course's requirements, each can be used as a basic scientific calculator and can solve linear and non-linear systems of equations and display results in graphical and/or tabular form. The authors do not emphasize or recommend any particular computer equation solver.

The introduction of modern equation solving tools into this course is meant to accomplish the following:

1. Motivate writing equations in symbolic form.
2. Save time by providing a rapid solution of (many) simultaneous equations.
3. Reduce computation error.
4. Provide a mathematical model of a problem that yields a fast and accurate means for a parametric study after appropriate testing for validity.
5. Provide a model for gaining a better understanding of the physics of the problem.
6. Stimulate interest and develop proficiency in the design process.

Design of a Circular Shaft Subjected to Combined Loading

Problem Statement

The problem presented has been simplified to demonstrate the proposed general structured procedure to solve machine design problems. A circular cross-section shaft rotates only a few degrees in service. The stress variation with time at any point in the shaft can be considered negligible; the safety factor can be determined with an appropriate static failure theory. The shaft is to be designed to support the loading shown in Figure 1. It is supported on self-aligning bearings (no bending resistance) at each end, and the left end bearing supports thrust (axial) loads.

The shaft is to be made of mild steel, and the yield strength, S_y, is known with a high degree of reliability from material testing laboratory tests. The radial loads P_B and P_C may experience an overload defined by κ_{P_B} and κ_{P_C}, respectively, the applied torques, Q_B and Q_C, overload factors are κ_{Q_B} and κ_{Q_C}, respectively, and the applied axial force, P_D, might be subject to an overload κ_{P_D} due to slight impact. Over and above the possible overload factors, we would like to have a design safety factor of N to account for other unknown factors.

In addition to the strength/stress factor of safety, we must limit the transverse deflection of the shaft to no more than δ_{max}. The effects of the compressive load on the lateral stiffness of the shaft is neglected to simplify the problem.

Develop the symbolic equations for designing a uniform shaft diameter to satisfy the factor of safety and deflection limitations.

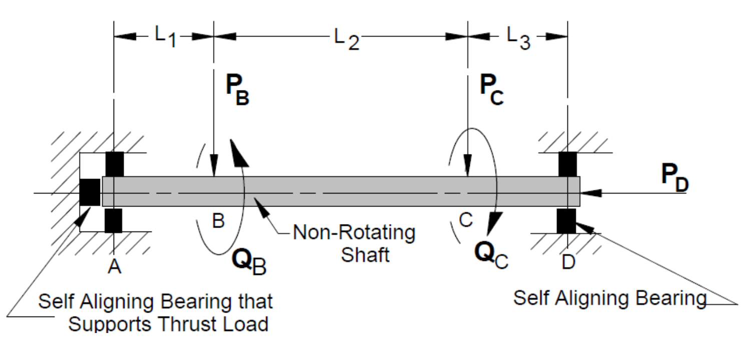

Figure 1. Proposed configuration and loading of a non-rotating machine shaft.

Using the symbolic model, calculate the shaft diameter for the following values of loading, geometry, overload factors, safety factors, and material.

$P_B = 300$ lb,	$P_C = 500$ lb,	$P_D = 1000$ lb	
$Q_B = 1500$ lb·in,	$Q_C = 1500$ lb·in		
$L_1 = 5$ in,	$L_2 = 10$ in,	$L_3 = 5$ in	
$\kappa_{P_B} = \kappa_{P_C} = 1.5$,	$\kappa_{P_D} = 4.0$,	$\kappa_{Q_B} = \kappa_{Q_C} = 1.3$,	$N = 2.0$
Mild Steel:	$S_y = 20\ (10^3)$ psi,	$E = 30\ (10^6)$ psi	
Deflection Limit:	$\delta_{max} = 0.005$ in		

Solution Phases for the Problem

A preliminary study of the shaft free-body diagram, shown as FBD I in Figure 2(a), confirms that the shaft reaction force system is statically determinate; there are three unknown reaction forces and three independent equilibrium equations. Consequently, equilibrium is sufficient to complete a force analysis without involving deformation formulas, compatibility, or boundary conditions. Therefore, the solution model development may be simplified by constructing the model in the following three phases.

- *Phase I - Equilibrium Analysis.* The complete force analysis will determine the internal forces and couples required to calculate stresses at a critical shaft cross-section. The critical cross-section will be identified.

- *Phase II - Stress Calculations and Failure Theory.* The shaft diameter will be determined to satisfy the strength limitation.

- *Phase III - Transverse Deflection Analysis.* The shaft diameter will be determined to satisfy the deflection limitation.

Selection of a standard size shaft diameter that satisfies both strength and deflection requirements will is required. It should be expected that the diameter resulting from Phase II would be different from Phase III, so satisfying both requirements will result in a diameter larger than necessary for one of the design requirements.

The detailed solution process for Phases I, II, and III, can be found in the Appendix. In each Phase, Step 8 Verify is addressed based on the solution required in each phase.

Conclusion

This paper presents a general structured procedure to solve machine design problems with a symbolic formulation approach to perform solution verification and parametric design studies. Teaching the student to model a general physical problem with the fundamental equations written in symbolic form, with no variable values specified, helps the student to more fully concentrate on the fundamental principles taught in the course. Introducing the modern engineering tool to solve the equations removes the necessary manipulation of the equations to isolate the dependent variables. Training the student to examine and test the answer becomes one essential goal in our course. The use of the symbolic approach, with mastery of a computer equation solver and the discipline to insist on verification, should be a significant asset in preparing students to model complex problems for analysis and design.

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Appendix: Phases I, II, and III for Design of a Circular Shaft Subjected to Combined Loading

Phase I - Equilibrium Analysis

The problem is solved using the structured approach previously discussed. The steps are defined as I-1, I-2, I-3, etc., where I denotes Phase I, and 1, 2, 3, etc., are the phase steps.

I-1 ***Model.*** To construct the symbolic solution, we will first develop equilibrium equations in terms of arbitrary variables to solve the support reactions and the internal shear forces and bending couples. The local XY coordinate system origin is located at the left end of the beam, as shown in Figure 2(a). Due to the shaft and loading discontinuities, three free-body diagrams, II, III, and IV, are required for a complete internal force analysis over the full length of the beam.

I-2 ***Free-Body Diagrams.*** Free-body diagrams are shown in Figure 2. Free-body diagram I is drawn to calculate the support reactions. The supports at A and D are smooth self-aligning bearings, which exert no bending or torsional couples. All unknown internal forces and couples have been assumed in their defined positive directions: normal force: tension, shear force: clockwise moment, and bending couples: positive curvature. Free-body diagrams II, III, and IV, define the internal force system within each of the three continuous loading regions, A-B, B-C, and C-D, respectively.

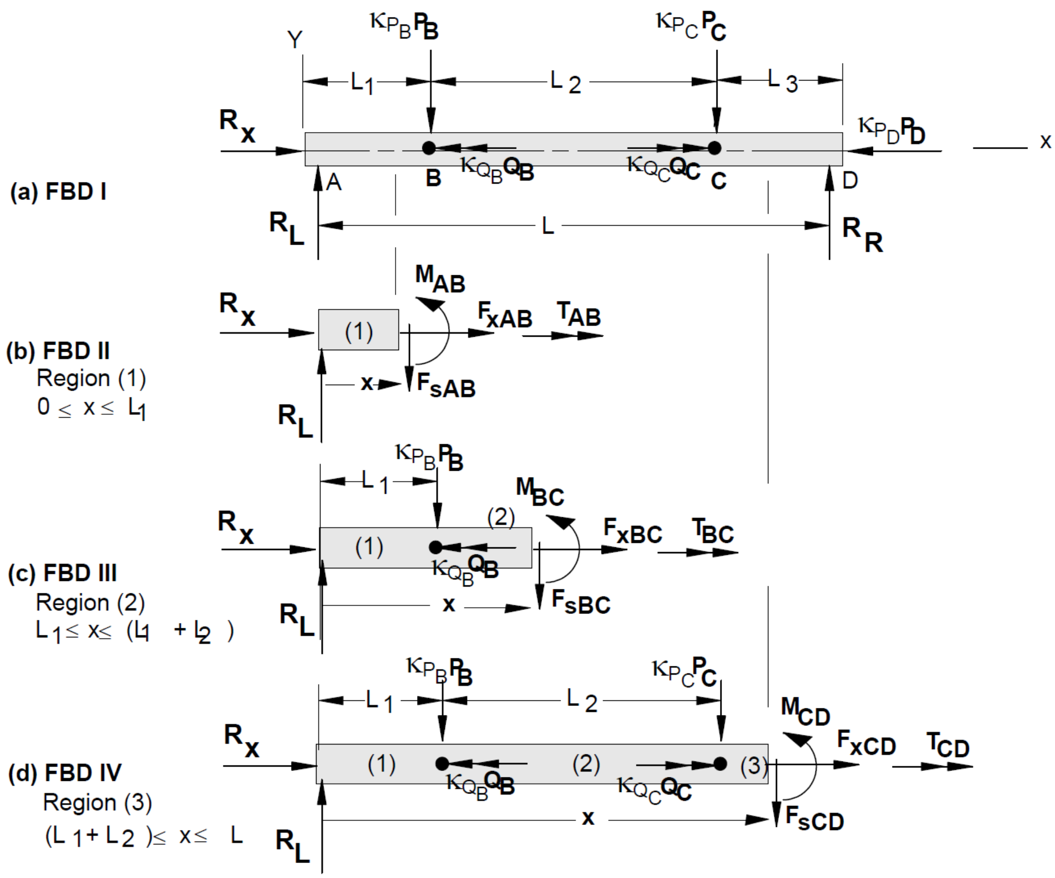


Figure 2. Free-body diagrams showing internal forces and couples in three continuous load regions.

I-3 ***Equilibrium Equations.*** Refer to the free-body diagrams in Figure 2. The external support reactions are found using FBD I.

$$\textit{FBD I,} \quad \sum F_x \quad : \quad R_x = \kappa_{P_D} P_D \qquad (1)$$

$$\sum F_Y \quad : \quad R_L + R_R = \kappa_{P_B} P_B + \kappa_{P_C} P_C \qquad (2)$$

$$\sum M_{z_D} \quad : \quad R_L\,(L) = \kappa_{P_B} P_B\,(L_2 + L_3) + \kappa_{P_C} P_C\,(L_3) \qquad (3)$$

where the factors κ_{P_D}, κ_{P_B}, and κ_{P_C} are the overload factors. The internal normal force, the shear force, bending and torsional couples throughout the complete length of the shaft can be found from discontinuous equilibrium equations from FBDs II, III, and IV:

$$
\begin{aligned}
&\textit{FBD II,} \quad 0 \leq x \leq L_1 \\
&\sum F_x : F_{xAB} + R_x = 0 && (4)\\
&\sum F_Y : F_{sAB} = R_L && (5)\\
&\sum M_{z_{cut}} : M_{AB} = R_L x && (6)\\
&\sum M_x : T_{AB} = 0 && (7)
\end{aligned}
$$

$$
\begin{aligned}
&\textit{FBD III,} \quad L_1 \leq x \leq L_1 + L_2 \\
&\sum F_x : F_{xBC} + R_x = 0 && (8)\\
&\sum F_Y : F_{sBC} + \kappa_{P_B} P_B = R_L && (9)\\
&\sum M_{z_{cut}} : M_{BC} + \kappa_{P_B} P_B (x - L_1) = R_L x && (10)\\
&\sum M_x : T_{BC} = \kappa_{Q_B} Q_B && (11)
\end{aligned}
$$

$$
\begin{aligned}
&\textit{FBD IV,} \quad L_1 + L_2 \leq x \leq L_1 + L_2 + L_3 \\
&\sum F_x : F_{xCD} + R_x = 0 && (12)\\
&\sum F_Y : F_{sCD} + \kappa_{P_B} P_B + \kappa_{P_C} P_C = R_L && (13)\\
&\sum M_{z_{cut}} : M_{CD} + \kappa_{P_B} P_B (x - L_1) + \kappa_{P_C} P_C (x - L_1 - L_2) \\
&\quad : \quad = R_L x && (14)\\
&\sum M_x : T_{CD} + \kappa_{Q_C} Q_C = \kappa_{Q_B} Q_B && (15)
\end{aligned}
$$

Remark. When building the solution to a problem, especially as the problem becomes complicated with many unknowns and governing equations, it is prudent to test and verify independent, smaller blocks of the solution. In this design problem, with a statically determinate force system, we will stop to verify the equilibrium analysis and then make a solution with the given input variables to identify the location and loading of the critical beam cross-section.
Had this problem been statically indeterminate, tests could be made by artificially, and temporarily remove indeterminacy. Still, there would be no choice for a true solution but to continue writing the deformation formulas to develop additional governing equations.

I-4 ***Deformation Formulas.*** Postponed until Phase III.

I-5 ***Compatibility and Boundary Conditions.*** Postponed until Phase III.

I-6 ***Complementary and Supporting Formulas.*** Postponed until Phase II.

I-7 ***Solve Phase I.*** There are 15 unknowns:

$$R_x,\ R_L,\ R_R,\ F_{xAB},\ F_{sAB},\ M_{AB},\ T_{AB}$$
$$F_{xBC},\ F_{sBC},\ M_{BC},\ T_{BC},\ F_{xCD},\ F_{sCD},\ M_{CD}\ \mathit{and}\ T_{CD}$$

which can be determined by a sequential solution of the 15 independent equilibrium equations. The problem is statically determinate. A plot of each internal force and couple should be made.

I-8 ***Verify Phase I.*** Before solving the equations with the given input specifications, test the solution of the programmed equations with the 'easy to check' input.

- Input the applied forces P_D, P_B, and P_C one at a time with various axial positions. Check the plotted results with hand calculation.

- Apply the torsional couples Q_B and Q_C at various axial positions and check the plotted results.

- Test results with different values of the κ load factors. Make sure that the correct load factor is applied to the particular applied load.

When satisfied with the verification, substitute the values of load and geometry specified in the Problem Statement. Then solve for and plot the axial distribution of the internal forces and couples in the shaft, as shown in Figure 3.

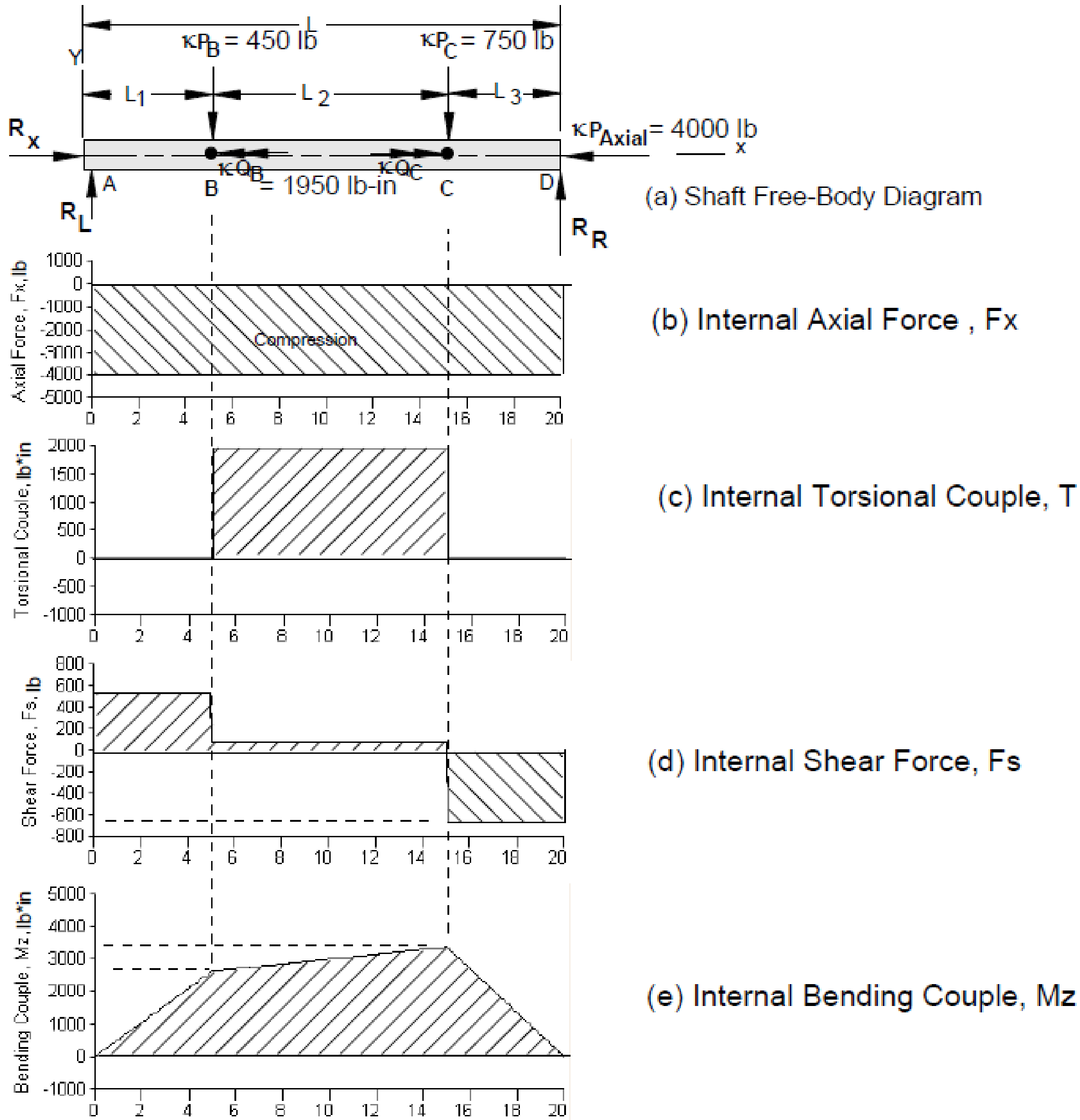


Figure 3. Internal force and couple diagrams for the shaft.

From a careful examination of the diagrams, we recognize that the critical cross-section must be at location B and/or C. Since the torsional couple is constant between B and C, and the axial load is constant between B and C, the critical cross-section depends on the magnitude of the bending couple at B and C. The critical section will be where the bending couple is maximum, in this case, location C. We can test this very simply in the program. For the data of this example, it appears that the cross-section subjected to the largest forces and couples is just to the left of point C, in the region (2) between locations B and C. However, in the general case where the applied forces and torsional couple can have any values, this may not always be the critical location, so we must carefully examine every new loading situation. Without question, however, the critical location will be at location B or C. We will now write equations to define the force systems at locations B and C, the force systems that are required for the stress calculations. These equations are derived directly from Equations 4 through 11 by merely specifying the appropriate value of the axial position x.

For location B, just to the left of point B in region (1), where $x = L_1$

$$
\begin{aligned}
F_{xB}^{(1)} &= F_{xAB}, \quad a\ constant && (16)\\
F_{sB}^{(1)} &= F_{sAB}, \quad a\ constant && (17)\\
M_{B}^{(1)} &= M_{AB}, \quad at\ x = L_1 \\
M_{B}^{(1)} &= R_L L_1 && (18)\\
T_{B}^{(1)} &= T_{AB}, \ a\ constant && (19)
\end{aligned}
$$

For location B, just to the right of point B in region (2), where $x = L_1$

$$
\begin{aligned}
F_{xB}^{(2)} &= F_{xBC}, \quad a\ constant && (20)\\
F_{sB}^{(2)} &= F_{sBC}, \quad a\ constant && (21)\\
M_{B}^{(2)} &= M_{B}^{(1)}, \quad at\ x = L_1 && (22)\\
T_{B}^{(2)} &= T_{BC}, \quad a\ constant && (23)
\end{aligned}
$$

For location C, just to the left of point C in region (2), where $x = L_1 + L_2$

$$
\begin{aligned}
F_{xC}^{(2)} &= F_{xBC}, \quad a\ constant && (24)\\
F_{sC}^{(2)} &= F_{sBC}, \quad a\ constant && (25)\\
M_{C}^{(2)} &= M_{BC}, \quad at\ x = L_1 + L_2 \\
M_{C}^{(2)} &= R_L\left(L_1 + L_2\right) - k_B P_B L_2 && (26)\\
T_{C}^{(2)} &= T_{BC}, \quad a\ constant && (27)
\end{aligned}
$$

For location C, just to the right of point C in region (3), where $x = L_1 + L_2$

$$
\begin{aligned}
F_{xC}^{(3)} &= F_{xCD}, \quad a\ constant && (28)\\
F_{sC}^{(3)} &= F_{sCD}, \quad a\ constant && (29)\\
M_{C}^{(3)} &= M_{C}^{(2)} && (30)\\
T_{C}^{(3)} &= T_{CD}, \quad a\ constant && (31)
\end{aligned}
$$

The solution of Equations 1 through 31 is straightforward and can be done sequentially because the problem is statically determinate. The equations were input to an equation solver program. For the specified load and geometry input, the solution yields the following results for the required internal force systems on the cross-sections in the region (2) side of locations B and C. The region (2) side at each location yields the maximum force system at B and C.

$$\begin{aligned} F_{xB}^{(2)} &= -4000\ lb,\quad F_{sB}^{(2)} = 75\ lb,\quad M_B^{(2)} = 2625\ lb\cdot\text{in},\quad T_B^{(2)} = 1950\ lb\cdot\text{in} \\ F_{xC}^{(2)} &= -4000\ lb,\quad F_{sC}^{(2)} = 75\ lb,\quad M_C^{(2)} = 3375\ lb\cdot\text{in},\quad T_C^{(2)} = 1950\ lb\cdot\text{in} \end{aligned}$$

Having tested the programmed equilibrium solution in the Verify Phase I step of the analysis, we can be confident in the result. Even so, a careful look at the results is a prudent step. Check the output values against the plotted values in Figure 3.

Phase II - Stress Calculations and Failure Theory

In Phase I, we derived the force system on the critical cross-section. We may now calculate the stress at the point of maximum stress on that cross-section. Then comparing the stress with the material yield strength through a failure theory, we will determine the required shaft diameter to yield the specified strength factor of safety.

We now apply the Step 6 Complementary and Supporting Formulas that were postponed in Phase I.

III-6 Complementary and Supporting Formulas.

(a) *Maximum Internal Force and Couples.* Figure 4 shows a typical cross-section in the shaft, not necessarily at B or C, with the particle coordinate system, the general force, and couple components, and the expressions for the combined uniform and flexure normal stress, σ_x, and the torsional shear stress τ_{xz}. The location of maximum stress on the selected cross-sections will be at m or n since, at those points, the flexure stress is maximum, the uniform normal stress is maximum, and the torsional shear stress is maximum. The two points should be tested because it is possible in another case that the axial load direction might be reversed. Recall that the flexure shear stress resulting from the shear force F_S is zero at the top and bottom points, m and n, of the cross-section (because of the area moment $Q_z = 0$). Relating the force components computed in Phase I to the symbols in the stress relationships shown in Figure 4, we have:

$$F_x = F_{xB}^{(2)} \qquad (32)$$

$$M_z = \max(M_B^{(2)}, M_C^{(2)}) \qquad (33)$$

$$T = T_B^{(2)} \qquad (34)$$

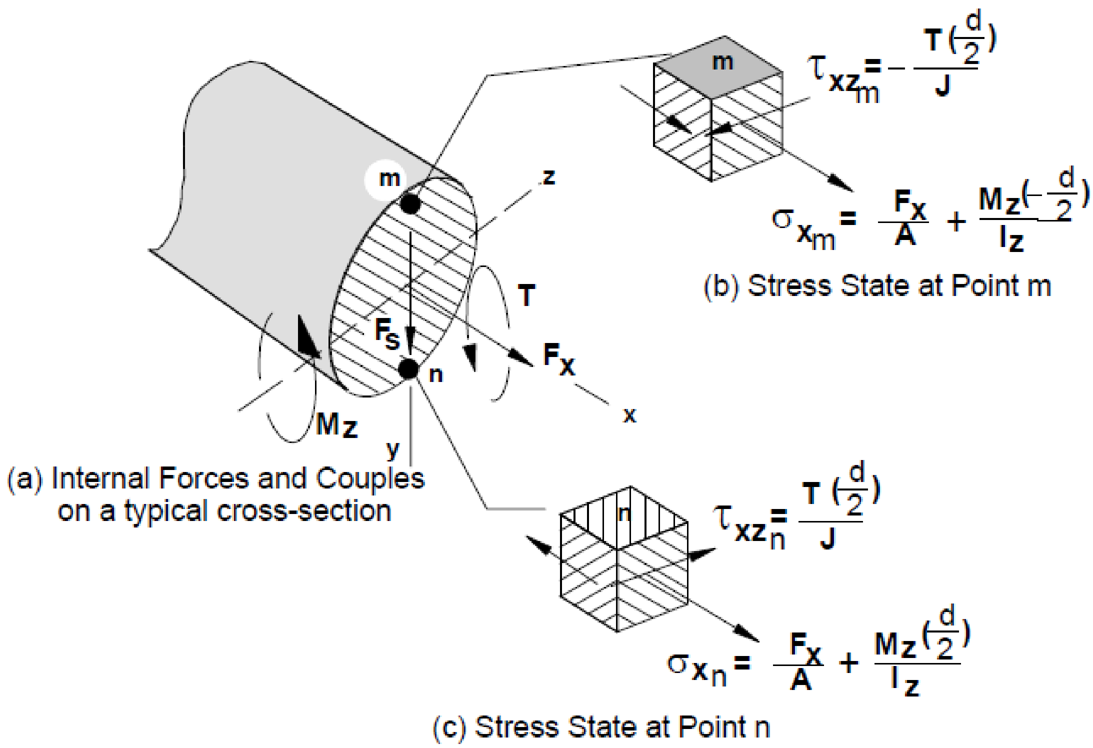


Figure 4. Stress calculation from internal normal and shear forces and torsional and bending couples.

(b) *Normal and Shear Stress Components.* The next step is to calculate the stress components at each point, m and n, on the critical cross-section,

Point m :

$$\sigma_{x_m} = \frac{F_x}{A} + \frac{M_z\left(-\frac{d}{2}\right)}{I_z} \tag{35}$$

$$\tau_{xz_m} = -\frac{T\left(\frac{d}{2}\right)}{J} \tag{36}$$

Point n :

$$\sigma_{x_n} = \frac{F_x}{A} + \frac{M_z\left(\frac{d}{2}\right)}{I_z} \tag{37}$$

$$\tau_{xz_n} = \frac{T\left(\frac{d}{2}\right)}{J} \tag{38}$$

where the area, A, moment of inertia, I_z, and polar moment of inertia, J, involve the unknown shaft radius.

(c) *Cross-Section Properties*.

i. *Cross-Section Area:*

$$A = \frac{\pi d^2}{4} \qquad \text{(i)}$$

ii. *Cross-Section Centroidal Area Moment of Inertia about the z axis:*

$$I_z = \frac{\pi d^4}{64} \qquad \text{(ii)}$$

iii. *Cross-Section Polar Moment of Inertia:*

$$J = \frac{\pi d^4}{32} \qquad \text{(iii)}$$

(d) *Failure Theory.* We elect to use the Distortion Energy Theory, preferable because it gives the best prediction of yield for ductile materials, plus there is no need to determine the principal stresses. To use the theory, we calculate the equivalent von Mises stress, σ' , from for each point, m and n,

$$\sigma' = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)}{2}}$$

Substituting the stress components which are zero,

$\sigma_y = 0 = \sigma_z = \tau_{xy} = \tau_{yz}$ at locations m and n

yields the following expressions for the von Mises stress at point m, σ'_m, and at point n, σ'_n;

$$\sigma'_m = \sqrt{(\sigma_{xm})^2 + 3(\tau_{xzm})^2} \qquad (39)$$

$$\sigma'_n = \sqrt{(\sigma_{xn})^2 + 3(\tau_{xzn})^2} \qquad (40)$$

Calculate and solve for the maximum of the two von Mises stress values:

$$\sigma'_{\max} = \max\left(\sigma'_m, \sigma'_n\right) \qquad (41)$$

and substitute into the definition of factor of safety:

$$N_{DET} = \frac{S_y}{\sigma'_{\max}} \qquad (42)$$

III-7 ***Solve Phase II.*** Equations 32 through 42 are appended to the equilibrium Equations 1 through 31 of Phase I. We have verified the solution of Equations 1 through 31. Now, Equations 32 through 38 must be solved and verified. Of course, none of the stress values can be calculated because the shaft's radius is unknown; it is to be determined. Here you must choose a solution procedure. How are you going to solve these equations? You can write them into an equation solver or substitute and try to construct one equation for the unknown radius as a function of the factor of safety. The latter is challenging. It involves a great deal of algebraic manipulation prone to error, and it still leads to a non-linear equation, which most likely would be solved by iteration.

Equations 1 through 31 for the force analysis were entered into an equation solver. We have elected to continue Equations 32 through 42 in the same equation solver. In using an equation solver program, to solve Equations 32 through 42, one approach would be to input diameters and develop a plot, as shown in Figure 5 of the resulting Distortion Energy Theory factor of safety versus the shaft diameter. According to the graph, a standard size shaft diameter of 1.75 in will meet the safety criterion.

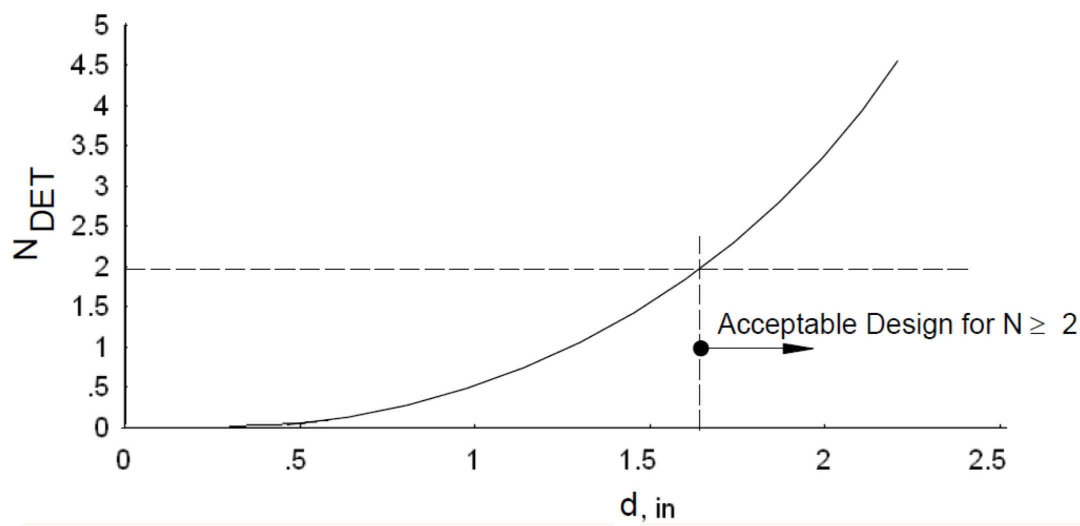


Figure 5. Factor of safety vs shaft diameter using the Distortion Energy Failure Theory.

A very convenient alternative approach can be taken using equation solver programs such as Matlab, Mathcad, or TKSolver. In these programs, the desired factor of safety can be input, and the diameter calculated directly with the built-in iterative procedure in these two programs.

III-8 ***Verify Phase II.*** Input a shaft diameter and run the solution.

- Compare the area properties with hand calculated values or/and other reference values.
- Study the signs of the stress components, check that the directions are physically possible.

- Examine the combination of flexure and uniform normal stress to check on the sign of each stress. Solve for the normal stress with opposite directions for P_D.

This step focused on verifying the stress calculations in Phase II. Each phase adds more complexity to the problem. After a new phase is added, the student should revisit the verify step for all previous phases to ensure the solution's confidence.

The next step is the investigation of the maximum deflection specification.

Phase III - Transverse Deflection Analysis

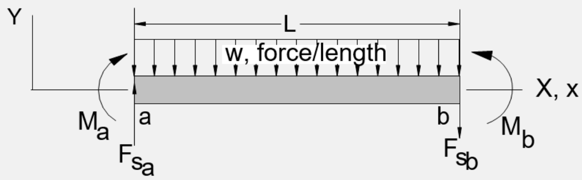
The transverse deflection of the shaft will occur only by the action of the transverse forces P_B and P_C. The axial load P_D and the torsional couples Q_B and Q_C do no affect on the transverse deformation.

We now apply Step 4 Deformation Formulas and Step 5 Compatibility and Boundary Conditions that were postponed in Phase I.

III-4 ***Deformation Formulas.*** The beam deformation formula for a prismatic, homogenous, and the uniformly loaded beam is used as shown Figure 6 and developed in [7]. To use the deformation formulas in Figure 6, the beam must be divided into segments, each having continuous uniform loading between their end points. Full segments are used to determine slope and displacement exactly at the end points of each full segment. To derive continuous displacement solutions throughout the length of the beam, we must cut each full segment into partial segments that have exposed cross-sections at arbitrary positions. The full and partial segments are shown in Figure 7. It is important to note that all of the forces on the FBDs in Figure 7 have been determined with Equations 1 through 27 of Phase I.

Apply beam deformation formulas to each of the partial and full segments (1), (2), and (3) shown in Figure 7.

Beam Deformation Formulas
Uniform Beam with End Loads
Plus Uniformly Distributed Load



$$\theta_b = \theta_a + \frac{M_b L}{EI_z} - \frac{F_{s_b} L^2}{2EI_z} - \frac{wL^3}{6EI_z}$$

$$v_b = v_a + \theta_a L + \frac{M_b L^2}{2EI_z} - \frac{F_{s_b} L^3}{3EI_z} - \frac{wL^4}{8EI_z}$$

Figure 6. Beam deformation formulas [7].

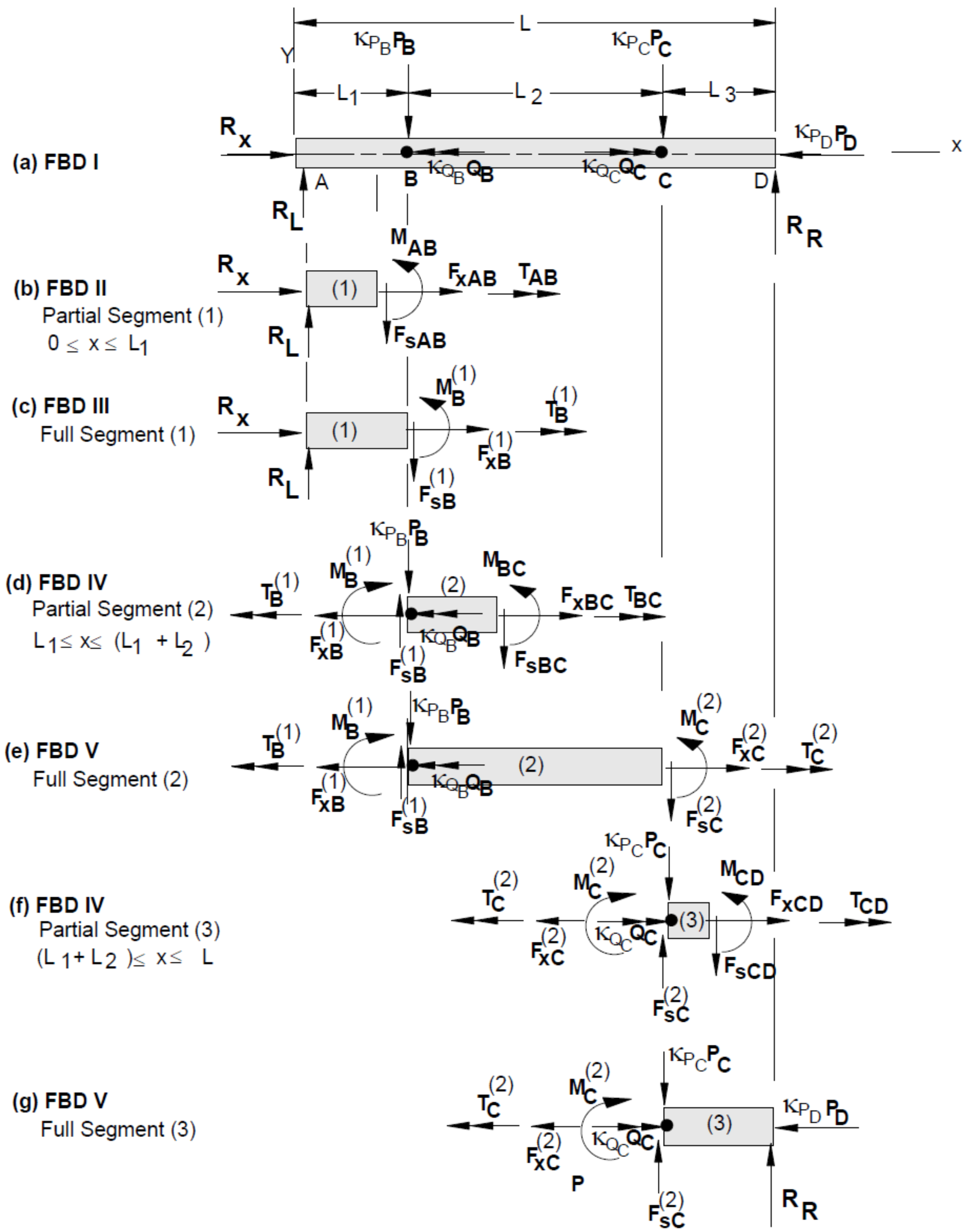


Figure 7. Free-body diagrams of full and partial segments of the shaft.

Partial Segment (1) from Point A to x:

$$\theta_{AB} = \theta_A + \frac{M_{AB}x}{EI_z} - \frac{F_{sAB}x^2}{2EI_z} \tag{43}$$

$$v_{AB} = v_A + \theta_A x + \frac{M_{AB}x^2}{2EI_z} - \frac{F_{sAB}x^3}{3EI_z} \tag{44}$$

Full Segment (1) from Point A to Point B:

$$\theta_B = \theta_A + \frac{M_B^{(1)}L_1}{EI_z} - \frac{F_{sB}^{(1)}L_1^2}{2EI_z} \tag{45}$$

$$v_B = v_A + \theta_A L_1 + \frac{M_B^{(1)}L_1^2}{2EI_z} - \frac{F_{sB}^{(1)}L_1^3}{3EI_z} \tag{46}$$

Partial Segment (2) from Point B to x:

$$\theta_{BC} = \theta_B + \frac{M_{BC}\left(x - L_1\right)}{EI_z} - \frac{F_{sBC}\left(x - L_1\right)^2}{2EI_z} \tag{47}$$

$$v_{BC} = v_B + \theta_B\left(x - L_1\right) + \frac{M_{BC}\left(x - L_1\right)^2}{2EI_z} - \frac{F_{sBC}\left(x - L_1\right)^3}{3EI_z} \tag{48}$$

Full Segment (2) from Point B to Point C:

$$\theta_C = \theta_B + \frac{M_C^{(2)}\left(L_2\right)}{EI_z} - \frac{F_{sC}^{(2)}\left(L_2\right)^2}{2EI_z} \tag{49}$$

$$v_C = v_B + \theta_B\left(L_2\right) + \frac{M_C^{(2)}\left(L_2\right)^2}{2EI_z} - \frac{F_{sC}^{(2)}\left(L_2\right)^3}{3EI_z} \tag{50}$$

Partial Segment (3) from Point C to x:

$$\theta_{CD} = \theta_C + \frac{M_{CD}\left(x - L_1 - L_2\right)}{EI_z} - \frac{F_{sCD}\left(x - L_1 - L_2\right)^2}{2EI_z} \tag{51}$$

$$v_{CD} = v_C + \theta_C\left(x - L_1 - L_2\right) + \frac{M_{CD}\left(x - L_1 - L_2\right)^2}{2EI_z} - \frac{F_{sCD}\left(x - L_1 - L_2\right)^3}{3EI_z} \tag{52}$$

Full Segment (3) from Point C to Point D:

$$\theta_D = \theta_C + \frac{R_R (L_3)^2}{2EI_z} \quad (53)$$

$$v_D = v_C + \theta_C (L_3) + \frac{R_R (L_3)^3}{3EI_z} \quad (54)$$

III-4 ***Compatibility and Boundary Conditions.*** The compatibility conditions of same slope and displacement at the junction of the segments are incorporated in the common label of the joined end points of each segment.

The boundary conditions are the zero displacement at the fixed end points A and D:

$$v_A = 0 \quad (i)$$

$$v_D = 0 \quad (ii)$$

III-6 ***Complementary and Supporting Formulas.*** None required.

III-7 ***Solution Phase III.*** Considering the boundary values, Equations i and ii as known, and with all forces solved in the equilibrium analysis, we have 12 unknowns:

$$\theta_{AB},\ \theta_{BC},\ \theta_{CD},\ \theta_A,\ \theta_B,\ \theta_C,\ \theta_D,\ v_{AB},\ v_{BC},\ v_{CD},\ v_B \text{ and } v_C$$

which will be solved over the full length of the shaft with Equations 43 through 54 appended to Equations 1 through 42 of Phases I and II. The design problem variables had been substituted into the equation solver program in Phase I and Phase II solutions. Trial values of standard diameters larger than the minimum required diameter of 1.75 in from Phase II were substituted until the computed maximum deflection did not exceed the requirement of 0.005 in. For a 2 in shaft diameter, the maximum shaft deflection was 0.0058 in which exceeded the limit. Going to the next standard size shaft diameter of 2.25 in, the maximum deflection of the beam was 0.0036 in, which is within the specification. Therefore, the design recommendation for the shaft diameter is 2.25 in. A plot of the shaft deflection for three different diameters is shown in Figure 8.

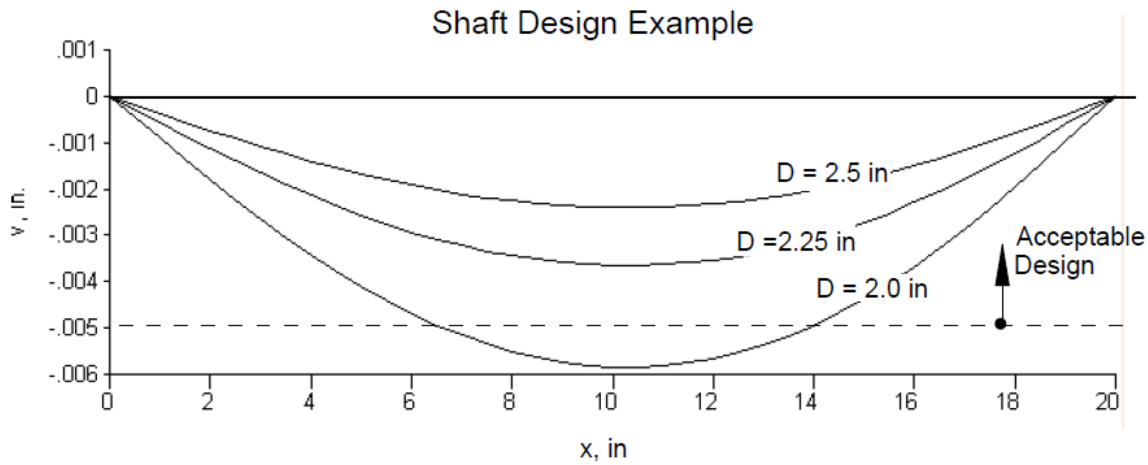


Figure 8. Transverse displacement of shaft for three trial shaft diameters.

Now, since the stress calculation suggested a shaft diameter between 1.5 in and 1.75 in, we will return to Figure 5 to check the distortion energy failure theory prediction for the new factor of safety with the larger shaft size of 2.25 in. As one would expect, the factor of safety is larger, in this case, close to 4.5, over twice that required in the design specification. The stiffness requirement has been the controlling factor in the design. This circumstance is common.

III-8 ***Verify Phase III.*** The programmed equations yield results, but are the results correct? Test the program.

- Check that the solution yields the maximum displacement slightly to the right of mid-span since $P_C > P_B$.

- Apply P_B = 1000 lb at mid-span with L_1 = 10 in and P_C = 0. For a 2 in diameter shaft, the maximum displacement should be

$$v_{\max} = \frac{P_B L^3}{48 E I_z} = \frac{\left(10^3\,\mathrm{lb}\right)\left(20\,\mathrm{in}\right)^3}{\left(48\right)30\left(10^6\right)\frac{\mathrm{lb}}{\mathrm{in}^2}\frac{\pi\left(2\,\mathrm{in}\right)^4}{64}} = 7.074\left(10^{-3}\right)\ \mathrm{in}$$

- Repeat with P_B = 0 and P_C = 1000 lb at mid-span with L_3 = 10 in.

As one final point of interest, we might ask what the overall factor of safety would have had to be in order to account for the force overloading factors. So, we go back to the program and make all overload factors unity and solve for the factor of safety. The distortion energy prediction calculates a factor of safety to be 8.0 compared to 4.5 for the diameter of 2.25 in. Interestingly, even with an overload factor of 4.0 on the axial force, its effect on the stress calculation was less than the other loads, so the overall increase in the design factor was slightly less than double. With a diameter of 2.25 in, a plot of the transverse displacement yields a curve shown in Figure 8. Values of slope and displacement at the locations A B, C, and D are found to be:

$$\begin{aligned}
\theta_A &= -5.71\left(10^{-4}\right)\,\mathrm{rad}, \qquad v_A = 0 \\
\theta_B &= -3.97\left(10^{-4}\right)\,\mathrm{rad}, \qquad v_B = -2.57\left(10^{-3}\right)\,\mathrm{in} \\
\theta_C &= 3.97\left(10^{-4}\right)\,\mathrm{rad}, \qquad v_C = -2.73\left(10^{-3}\right)\,\mathrm{in} \\
\theta_D &= 6.21\left(10^{-4}\right)\,\mathrm{rad}, \qquad v_D = 0
\end{aligned}$$

The displacement plot and/or a tabulated list of transverse displacement over the full length of the shaft yield the location a nd value of maximum deflection as:

$$x = 10.3\,\mathrm{in}, \qquad \theta = 0, \qquad v_{\max} = -3.65\left(10^{-3}\right)\,\mathrm{in}$$

This step focused on verifying the transverse deflection analysis in Phase III. As previously discussed, each phase adds more complexity to the problem. After a new phase is added, the student should revisit the verify step for all previous phases to ensure the solution’s confidence.