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A Direct Method of Determining the Natural Frequency and Dimensionless Damping Coefficient of any Second-order Circuit

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Abstract

All electrical and computer engineering students, and those from many other engineering disciplines, learn to generate first- and second-order ordinary linear differential equations and their solutions from circuit diagrams by applying basic analytic methods and laws, such as Kirchhoff's laws and Ohm's law. These differential equations and their solutions are expressed in terms of key characteristic parameters: a time constant for first-order circuits and the dimensionless damping coefficient and natural frequency for second-order circuits. Unfortunately, methods of designing a circuit to achieve desired values of these parameters are generally not taught and consequently are less well understood.

A method is presented in this paper that facilitates the analysis and design of a second-order circuit with prescribed characteristics. The method extends a fundamental result of Thévenin's theorem to relate four time constants to the dimensionless damping coefficient and natural frequency. These time constants are readily associated with simple first-order circuit structures that act as guides in the construction of a circuit with desired characteristics. This method has the added pedagogical advantage of offering students an alternate and potentially appealing means of generating these characteristic parameters from direct observation of a second-order circuit.

Preliminary assessment data suggests improved problem solving for both first- and second-order circuit analysis. Across 4 different problems solved by 19 students there were 28 cases in which students were unable to correctly solve a problem using traditional methods, while in 17 of those 28 cases the students were able to do so using the approach outlined in this paper. There were no instances in which a student was successful using the traditional approach but unsuccessful using this new approach. All students received instruction in both methods.

Introduction

The transient (homogeneous) solution of any first-order system with constant parameters is described by the following expression.

$$Ke^{-t/\tau} \tag{1}$$

where τ is the time constant associated with the system and K is related to an initial condition. In the cases of first-order RC and RL circuits, the time constants are

$$\tau = R_{\text{T}}C \qquad \text{and} \qquad \tau = L/R_{\text{N}} \tag{2}$$

where R_{T} and R_{N} are the Thévenin and Norton equivalent resistances seen by a capacitor C or an inductor L, respectively. Methods for determining R_{T} and R_{N} are well-known and generally well-understood by students. As a result, it is not necessary or perhaps even constructive to derive and solve a differential equation when teaching students about first-order system behavior, analysis and design.

On the other hand, a similar second-order RLC system has three possible transient solutions: overdamped, critically damped and underdamped. These solutions depend upon a natural frequency ω_0 and a dimensionless damping coefficient ζ. The value of ζ determines the specific solution. The relationship between ζ and RLC circuit parameters is given by

$$\frac{2\zeta}{\omega_0} = R_{\text{T}}C + \frac{L}{R_{\text{N}}} \tag{3}$$

Here, R_{T} is the Thévenin equivalent resistance seen by the capacitor when the inductor is treated as a short-circuit, which is its dc equivalent or its equivalent when $L = 0$. Likewise, R_{N} is the Norton equivalent resistance seen by the inductor when the capacitor is treated as an open-circuit, which is its dc equivalent or its equivalent when $C = 0$. However, except for the cases of series LC and parallel LC circuits, a general relationship between ω_0 and RLC circuit parameters is not found in textbooks [1] and is not generally well-understood. As a result, students are taught to derive a second-order differential equation for each instance of a second-order circuit in order to determine ω_0 and subsequently ζ. This tedious process provides specific solutions to specific circuits but is often confusing to students and generally does not result in a broader, general understanding of the relationship between second-order transient behavior and RLC circuit parameters.

The result described in this paper is a direct, comprehensible relationship between second-order transient behavior and RLC circuit parameters. As shown in the next section, that relationship involves time constants and the concept of Thévenin/Norton equivalent resistances seen by capacitors and inductors similar to those found in first-order circuits.

Literature

The subject of Thévenin/Norton equivalent networks was explored and developed thoroughly over a roughly one hundred year period from 1850 to 1950. The underlying theorem was first published in 1853 by Hermann von Helmholtz and independently and later by Leon Charles Thévenin in 1883. Johnson [2] reviewed the early development of what he termed the equivalent voltage-source theorem. Later, in 1926, Hans Ferdinand Mayer and Edward Lawry Norton independently published what Johnson [3] termed the equivalent current-source theorem.

More recent work has focused on alternatives to [4], extensions of [5], [6], [7], [8] and variations in the methods used to apply [9] and teach [10] these theorems. Unfortunately, no recent papers have yet been found that directly relate to the subject of this paper, which is the relationship between fundamental transient circuit behavior and the concept of Thévenin/Norton equivalent resistances. This result is likely due to the subject being well-established and largely settled.

A New Method

The method presented here does not require the explicit derivation of an ODE. Instead, it relies on Thévenin's theorem to derive low and high frequency time constants, which are used to determine ζ and ω_0. In general, the standard form of a homogeneous second-order differential equation with constant coefficients is

$$\frac{1}{\omega_o^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_0}\frac{dx}{dt}+x=0 \tag{4}$$

The coefficient of the first-order derivative has a dimension of time and can be expressed in terms of two time constants here designated as τ_C^{short} and τ_L^{open}, such that

$$\frac{2\zeta}{\omega_0}=\tau_C^{\text{short}}+\tau_L^{\text{open}} \tag{5}$$

where

$$\tau_C^{\text{short}}=R_{\text{T}}C \qquad \text{and} \qquad \tau_L^{\text{open}}=L/R_{\text{N}} \tag{6}$$

and R_{T} is the Thévenin equivalent resistance seen by the capacitor when the inductor is treated as a short-circuit and R_{N} is the Norton equivalent resistance seen by the inductor when the capacitor is treated as an open-circuit. The notation used here for the two time constants reflects those statements. These treatments are well-known low frequency approximations.

The coefficient of the second-order derivative has a dimension of time squared, which suggests a possible alternative representation as the product of two time constants τ_1 and τ_2.

$$\frac{1}{\omega_o^2}=\tau_1\tau_2 \tag{7}$$

The question is whether these arbitrary time constants can be related, in general, to RLC circuit parameters. It is asserted here, without proof, that they can! Although no formal proof is offered, it can be demonstrated that the product of those two constants has two equivalent representations.

$$\tau_1\tau_2=\tau_C^{\text{short}}\tau_L^{\text{short}} \qquad \text{or} \qquad \tau_1\tau_2=\tau_C^{\text{open}}\tau_L^{\text{open}} \tag{8}$$

where τ_C^{short} and τ_L^{open} are as defined above. For the purposes of what follows, these time constants are better expressed by an expanded notation as

$$\tau_C^{\text{short}}=R_{\text{T}}^{\text{short}}C \qquad \text{and} \qquad \tau_L^{\text{open}}=L/R_{\text{N}}^{\text{open}} \tag{9}$$

The definitions of the other two time constants τ_C^{open} and τ_L^{short} can be inferred from the notation.

$$\tau_C^{\text{open}}=R_{\text{T}}^{\text{open}}C \qquad \text{and} \qquad \tau_L^{\text{short}}=L/R_{\text{N}}^{\text{short}} \tag{10}$$

Here, the capacitor and inductor are treated as short- and open-circuits, respectively, in the calculation of $R_{\mathrm{N}}^{\mathrm{short}}$ and $R_{\mathrm{T}}^{\mathrm{open}}$. These treatments are well-known high frequency approximations.

To summarize and clarify, each time constant depends upon a Thévenin/Norton equivalent resistance calculation or measurement at a very low or high frequency, where

- $R_{\mathrm{T}}^{\mathrm{short}}$ indicates the Thévenin equivalent resistance seen by the capacitor with the inductor treated as a short-circuit.
- $R_{\mathrm{N}}^{\mathrm{open}}$ indicates the Norton equivalent resistance seen by the inductor with the capacitor treated as an open-circuit.
- $R_{\mathrm{T}}^{\mathrm{open}}$ indicates the Thévenin equivalent resistance seen by the capacitor with the inductor treated as an open-circuit.
- $R_{\mathrm{N}}^{\mathrm{short}}$ indicates the Norton equivalent resistance seen by the inductor with the capacitor treated as a short-circuit.

Two of the four time constants, τ_C^{short} and τ_L^{open}, are necessary along with one of the other two, τ_C^{open} and τ_L^{short}, to determine the dimensionless damping coefficient ζ and the natural frequency ω_0, as summarized below.

$$\frac{2\zeta}{\omega_0} = \tau_C^{\mathrm{short}} + \tau_L^{\mathrm{open}} \tag{5}$$

$$\frac{1}{\omega_o^2} = \tau_C^{\mathrm{short}}\tau_L^{\mathrm{short}} = \tau_C^{\mathrm{open}}\tau_L^{\mathrm{open}} \tag{7-8}$$

Special Case: Series LC

In the special case of a series LC circuit, the Thévenin/Norton resistances reduce to a pair of values, one of which is unbounded.

$$R_{\mathrm{T}}^{\mathrm{open}} = R_{\mathrm{N}}^{\mathrm{open}} \to \infty \qquad \text{and} \qquad R_{\mathrm{T}}^{\mathrm{short}} = R_{\mathrm{N}}^{\mathrm{short}}$$

The resulting time constants are

$$\tau_C^{\mathrm{short}} = R_{\mathrm{T}}^{\mathrm{short}}C \qquad \text{and} \qquad \tau_L^{\mathrm{open}} = L/R_{\mathrm{N}}^{\mathrm{open}} \to 0 \tag{11}$$

and

$$\tau_C^{\mathrm{open}} = R_{\mathrm{T}}^{\mathrm{open}}C \to \infty \qquad \text{and} \qquad \tau_L^{\mathrm{short}} = L/R_{\mathrm{N}}^{\mathrm{short}} \tag{12}$$

such that

$$\frac{2\zeta}{\omega_0} = \tau_C^{\mathrm{short}} + \tau_L^{\mathrm{open}} = \tau_C^{\mathrm{short}} = R_{\mathrm{T}}^{\mathrm{short}}C \tag{13}$$

and

$$\frac{1}{\omega_o^2} = \tau_C^{\mathrm{short}}\tau_L^{\mathrm{short}} = R_{\mathrm{T}}^{\mathrm{short}}C\frac{L}{R_{\mathrm{N}}^{\mathrm{short}}} = LC \tag{14}$$

It is worth noting that the product of the other pair of time constants $\tau_C^{\mathrm{open}}\tau_L^{\mathrm{open}}$ is finite but indeterminate.

Special Case: Parallel LC

In the special case of a parallel LC circuit, the Thévenin/Norton resistances again reduce to a pair of values, one of which is unbounded.

$$R_{\text{T}}^{\text{open}} = R_{\text{N}}^{\text{open}} \qquad \text{and} \qquad R_{\text{T}}^{\text{short}} = R_{\text{N}}^{\text{short}} = 0$$

The resulting time constants are

$$\tau_C^{\text{short}} = R_{\text{T}}^{\text{short}} C = 0 \qquad \text{and} \qquad \tau_L^{\text{open}} = L/R_{\text{N}}^{\text{open}} \tag{15}$$

and

$$\tau_C^{\text{open}} = R_{\text{T}}^{\text{open}} C \qquad \text{and} \qquad \tau_L^{\text{short}} = L/R_{\text{N}}^{\text{short}} \to \infty \tag{16}$$

such that

$$\frac{2\zeta}{\omega_0} = \tau_C^{\text{short}} + \tau_L^{\text{open}} = \tau_L^{\text{open}} = \frac{L}{R_{\text{N}}^{\text{open}}} \tag{17}$$

and

$$\frac{1}{\omega_o^2} = \tau_C^{\text{open}} \tau_L^{\text{open}} = R_{\text{T}}^{\text{open}} C \frac{L}{R_{\text{N}}^{\text{open}}} = LC \tag{18}$$

It is worth noting that the product of the other pair of time constants $\tau_C^{\text{short}} \tau_L^{\text{short}}$ is again finite but indeterminate.

Special Case: First-order RC

When $L = 0$ the system of equations should yield the result for a first-order RC circuit. In this case, the time constants are

$$\tau_C^{\text{short}} = R_{\text{T}}^{\text{short}} C \qquad \text{and} \qquad \tau_L^{\text{open}} = L/R_{\text{N}}^{\text{open}} = 0 \tag{19}$$

and

$$\tau_C^{\text{open}} = R_{\text{T}}^{\text{open}} C \qquad \text{and} \qquad \tau_L^{\text{short}} = L/R_{\text{N}}^{\text{short}} = 0 \tag{20}$$

such that the products in Equation (8) are both zero. The result is that the coefficients of the first- and second-derivative terms in Equation (4) are $R_{\text{T}} C$ and zero, respectively, yielding the correct first-order homogeneous differential equation.

$$R_{\text{T}} C \frac{dx}{dt} + x = 0 \tag{21}$$

Special Case: First-order RL

When $C = 0$ the system of equations should yield the result for a first-order RL circuit. In this case, the time constants are

$$\tau_C^{\text{short}} = R_{\text{T}}^{\text{short}} C = 0 \qquad \text{and} \qquad \tau_L^{\text{open}} = L/R_{\text{N}}^{\text{open}} \tag{22}$$

and

$$\tau_C^{\text{open}} = R_{\text{T}}^{\text{open}} C = 0 \qquad \text{and} \qquad \tau_L^{\text{short}} = L/R_{\text{N}}^{\text{short}} \tag{23}$$

such that the products in Equation (8) are again both zero. The result is that the coefficients of the first- and second-derivative terms in Equation (4) are L/R_{N} and zero, respectively, yielding the correct first-order homogeneous differential equation.

$$\frac{L}{R_{\text{N}}}\frac{dx}{dt} + x = 0 \tag{24}$$

Example Problem

A typical second-order LC example problem that illustrates the value of the new method is shown in Figure 1. Note that the inductor and capacitor are not in series or parallel. As a result, it is not obvious how the natural frequency of the network is related to the circuit parameters.

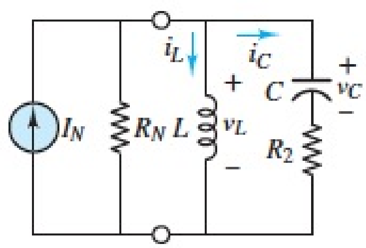

Figure 1: Example second-order circuit

The traditional approach to the type of second-order circuit problem shown in Figure (1) is to apply Kirchhoff's laws twice to ultimately derive a second-order differential equation. In this example, Kirchhoff's current law (KCL) applied to the upper node yields

$$I_{\text{N}} - \frac{v_L}{R_{\text{N}}} - i_L - i_C = I_{\text{N}} - \frac{v_L}{R_{\text{N}}} - i_L - C\frac{dv_C}{dt} = 0 \tag{25}$$

where the differential constitutive relationship for the capacitor $i_C = C dv_C/dt$ is used to replace i_C. Also, Kirchhoff's voltage law (KVL) applied around the far right mesh yields

$$v_L - v_C - i_C R_2 = L\frac{di_L}{dt} - v_C - R_2 C\frac{dv_C}{dt} = 0 \tag{26}$$

where the differential constitutive relationships for the capacitor $i_C = C dv_C/dt$ and for the inductor $v_L = L di_L/dt$ are used to replace i_C and v_L, respectively. The result is two first-order differential equations in the two unknown state variables i_L and v_C.

The process so far is not overly difficult and one of these two first-order equations is needed to include initial conditions in the complete solution. However, students often find the process of combining equations (25) and (26) to eliminate one of the state variables confusing, difficult and prone to errors, especially sign errors. It is left as an exercise for the reader to work through the details. If done correctly the homogeneous form of the resulting second-order differential equation is

$$\frac{R_{\text{N}} + R_2}{R_{\text{N}}} LC \frac{d^2x}{dt^2} + \left(R_2C + \frac{L}{R_{\text{N}}} \right) \frac{dx}{dt} + x = 0 \tag{27}$$

where x is any variable in the network. The natural frequency ω_0 and dimensionless damping coefficient ζ are found by comparing equation (27) to the standard form previously presented in equation (4) and repeated here for convenience.

$$\frac{1}{\omega_0^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_0} \frac{dx}{dt} + x = 0 \tag{4}$$

The alternate method proposed in this paper finds the four equivalent resistances and their associated time constants shown in equations (9) and (10), which are then used to determine directly the natural frequency ω_0 and the dimensionless damping coefficient ζ. For the example problem, equations (9) and (10) become

$$\tau_C^{\text{short}} = R_{\text{T}}^{\text{short}} C = R_2 C \qquad \text{and} \qquad \tau_L^{\text{open}} = L/R_{\text{N}}^{\text{open}} = L/R_{\text{N}} \tag{28}$$

and

$$\tau_C^{\text{open}} = R_{\text{T}}^{\text{open}} C = (R_{\text{N}} + R_2)C \qquad \text{and} \qquad \tau_L^{\text{short}} = L/R_{\text{N}}^{\text{short}} = L/(R_{\text{N}} \parallel R_2) \tag{29}$$

These time constants can be substituted into equations (5), (7) and (8) to yield

$$\frac{2\zeta}{\omega_0} = R_2C + L/R_{\text{N}} \tag{30}$$

and

$$\frac{1}{\omega_o^2} = R_2C \frac{L}{R_{\text{N}} \parallel R_2} = \frac{R_{\text{N}} + R_2}{R_{\text{N}}} LC \tag{31}$$

or

$$\frac{1}{\omega_o^2} = (R_{\text{N}} + R_2)C \frac{L}{R_{\text{N}}} = \frac{R_{\text{N}} + R_2}{R_{\text{N}}} LC \tag{32}$$

Of course, these results are identical to those obtained through the derivation of the second-order differential equation. It is suggested here that the new alternate method is more efficient and less prone to error. It is also suggested that the relationship between the circuit parameters and the circuit behavior, as expressed by the natural frequency and dimensionless damping coefficient, is clarified by the alternate method because that behavior is now linked explicitly to the Thévenin/Norton equivalent resistances seen by the capacitor and inductor.

Pedagogical Implications

One obvious result of the method outlined in this paper is that an alternative to the traditional differential equation approach exists for the analytic problem solving of transient circuit behavior. This method relies only on basic circuit analysis skills (e.g., the ability to determine the equivalent resistance between two nodes) and concepts (e.g., time constants) and thereby permits students to focus on and better comprehend the relationship between the new concepts of damping and natural frequency and equivalent resistance, capacitance and inductance. This method when applied to problem solving also involves fewer algebraic manipulations, which are the source of many errors in student work. This latter benefit is suggested by the initial assessment data.

This direct relationship between equivalent resistance, damping and natural frequency also encourages an intuitive understanding of transient behavior and may lower the barrier to applications of transient circuit analysis in engineering design. It is in the realm of electrical engineering design that it is hoped this method may prove particularly useful. Of course, powerful numerical and symbolic methods exist for generating data associated with complex physical systems. However, those methods rely on accurate models and it is in this regard that it is hoped the method outlined in this paper may yield some benefit through the simpler and more direct relationship between equivalent resistance and transient behavior.

Initial Assessment of Pedagogical Efficacy

This new approach as well as the differential equation approach found in nearly all textbooks were used in the spring 2020 to teach analytic problem solving of second-order transient circuits to one group of 19 students, all of whom were sophomore electrical or computer engineering majors. The students were presented with 4 different second-order transient circuit problems on two graded exams. The problems required the students to find only the dimensionless damping coefficient and the natural frequency. The circuits on each exam were technically identical except for differences in parameter values and aesthetic appearance. Care was taken to ensure that the changes in appearance did not result in greater or lesser apparent complexity. The exam requiring the derivation of differential equations was given three days prior to the exam requiring the use of the proposed new method. Of the total of 76 solutions, 48 were done correctly using the traditional differential equation approach while 65 were done correctly using the new approach described in this paper. Furthermore, an anonymous poll of the students taken one week after the conclusion of the exam indicated that 10 of the 19 students preferred the new approach, 5 students had no preference and 4 preferred the differential equation approach.

	Traditional Method	New Method
Total # of Problems	76	76
Total # Solved Correctly	48	65
Percentage Solved Correctly	63%	85%

Table 1: Initial results of second-order transient circuit problem solving by sophomore electrical and computer engineering majors.

These initial results are preliminary and not definitive. No effort was made at the time to examine and compare the sources of error in the student solutions. Additional and more carefully controlled studies are needed to assess the pedagogical efficacy of the new approach.

Conclusion and Future Work

The likely principal pedagogical advantage of this method is its focus on the role played by Thévenin/Norton equivalent resistances in determining the behavior of first- and second-order circuits. The simplicity of those concepts may aid students to more easily design practical circuits. The method may also enable students to more easily develop an intuitive understanding of physical systems through a better understanding of practical discrete models of those systems. Practical models of electrolytic capacitors and inductive windings are two good examples of such models. All of these possibilities remain to be explored, developed and assessed.

There is also work to be done in understanding the physical significance and implications of the low- and high-frequency approximations employed in the method.

Further development of the outlined method is needed to determine its value, in general, and pedagogical efficacy, in particular. The next step is to show that the method also accounts for second-order circuits with two capacitors and with two inductors. It is believed that the method can be scaled to higher-order networks but that result has yet to be demonstrated.

References

[1] G. Rizzoni and J. Kearns. Principles and Applications of Electrical Engineering, 7th ed., McGraw-Hill, 2021, pp. 257-316.

[2] D.H. Johnson, Origins of the equivalent circuit concept: the voltage-source equivalent. Proceedings of the IEEE, 91.4, 2003, pp. 636-640.

[3] D.H. Johnson. Origins of the equivalent circuit concept: The current-source equivalent. Proceedings of the IEEE, 91.5, 2003, pp. 817-821.

[4] R. Hashemian, Hybrid equivalent circuit, its properties and application, and alternative to Thévenin and Norton equivalents, Proceedings of the Midwest Symposium On Circuits and Systems, MWSCAS, 800-803, 2009.

[5] R. E. Bryant, J. D. Tygar, and L. P. Huang. Geometric characterization of series-parallel variable resistor networks, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 41.11, 1994, pp. 686-698.

[6] A. Penin, About the definition of parameters and regimes of active two-port networks with variable loads on the basis of projective geometry. WSEAS Transactions on Circuits and Systems, Vol.10, No.5, 2011, pp. 157-172.

[7] A. Penin, Generalized Thévenin/Helmholtz and Norton/Mayer Theorems of Electric Circuits with Variable Resistances, WSEAS Transactions on Circuits and Systems, Vol.13, No.5, 2014, pp. 104-116.

[8] G. Mihai. Generalisation of Helmholtz-Thevenin theorem to three-phase electrical circuits, SITECH, June, 2009.

[9] G. Chatzarakis. A New Method for Finding the Thévenin and Norton Equivalent Circuits, Engineering 2(5): 328-336, 2010.

[10] J. Vandewalle, Shortcuts in circuits and systems education with a case study of the Thévenin/Helmholtz and Norton/Mayer equivalents, IEEE International Symposium on Circuits and Systems (ISCAS), 2012.